*Normed Linear Space 1.1:* Let *X* be a vector space over either the scalar field of real numbers or the scalar field of complex numbers. Suppose we have a function

: *X →* [0*,*∞) such that

(1) = 0 if and only if *x* = 0, *(definiteness)*

(2) ≤ + for all *x, y ∈ X*,

 *(triangle inequality)* and

(3) = |α| for all scalars α and vectors *x*.

 *(homogeneity) MacCluer,* p. 2.

*Metric Space 1.9:* A *metric space* is a set *X* with a function *d*(*·, ·*) : *X ×X →* [0*,*∞) satisfying, for *x,y,* and

*z* in *X*,

(1) *d*(*x,y*) = 0 if and only if *x* = *y*, (definiteness)

(2) *d*(*x,y*) = *d*(*y,x*), and (symmetry)

(3) *d*(*x,y*)+*d*(*y, z*) *≥ d*(*x, z*). (triangle inequality)

 *MacCluer,* p. 5.

*Cauchy Sequence 1.10:* Let *X* be a metric space. A sequence *{xn}* in *X* is said to be a *Cauchy sequence* if it has the following property: Given any ε *>* 0 there exists *N* such that if *n, m ≥ N*, then *d*(*xn, xm*) *<* ε .

 *MacCluer,* p. 5.

*Complete Metric Space 1.11:* A metric space is said to be *complete* if every Cauchy sequence in *X* converges in *X*.

 *MacCluer,* p. 5.

Cauchy Sequence 1.10

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Complete Metric Space 1.11

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Normed Linear Space 1.1

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Metric Space 1.9

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*Banach space 1.12:* Let *X* be a normed linear space. If *X* is complete in the metric *d* defined from the norm by *d*(*x,y*) =, we call *X* a *Banach space*.

*MacCluer,* p. 5.

*Inner Product 1.13:* Let *X* be a vector space over C. An *inner product* is a map : *X ×X →* satisfying, for

*x, y*, and *z* in *X* and scalars α *∈* ,

(1)  = for all *x, y* in *X*,

 *(hermitean, denotes complex conjugation.)*

(2) ≥0, with = 0 (if and) only if *x* = 0,

 *(positive-definiteness)*

(3) =  +, and

(4) = α. ( 3 & 4 together = sequilinearity)

 *MacCluer,* p. 6.

*Proposition 1.14:* If is an inner product on a vector space *X*, then for all *x* and *y* in *X* we have

  ≤ *.*

 *MacCluer,* p. 7.

*Proposition 1.15:* If is an inner product on a vector space *X,* then

 *MacCluer,* p. 7.

Proposition 1.14

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Proposition 1.15

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Banach space 1.12

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Inner Product 1.13

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*Hilbert space 1.16:* A (complex) *Hilbert space* is a vector space over with an inner product such that is complete in the metric

 *d*(*x, y*) = =*.*

*MacCluer,* p. 8.

*Proposition 1.18:* If f is a analytic function in some closed disk *B*(*a,R*)*,* then

 *MacCluer,* p. 9.

*Corollary 1.19:* Fix *w ∈* D*.* For every

 we have

 *MacCluer,* p. 9.

*Theorem 1.20:* The Bergman space is a Hilbert space.

 *MacCluer,* p. 10.

Corollary 1.19

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Theorem 1.20

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Hilbert space 1.16

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Proposition 1.18

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*Orthogonality 1.21:* Given vectors *f ,g* in a Hilbert space , we say that *f* is *orthogonal* to *g*, written *f ⊥ g*, if

= 0. For sets *A* and *B* in we write *A* *⊥ B* if

= 0 for all *f ∈ A* and *g ∈ B*. Finally, is the set of all vectors *f ∈* such that *f ⊥ g* for all *g* in *A*; for any set *A* this is always a subspace of , moreover since

= , is a closed subspace by continuity of the inner product (see Exercise 1.8).

 *MacCluer,* p. 11.

*Proposition 1.22:* If , are pairwise orthogonal vectors in a Hilbert

space, then

*.*

 *MacCluer,* p. 11.

*Proposition 1.23:* Every nonempty, closed convex set *K* in a Hilbert spacecontains a unique element of smallest norm. Moreover, given any *h ∈ ,* there is a unique in *K* such that

*.*

 *MacCluer,* p. 12.

*Theorem 1.24:* Let *M* be a closed subspace of a Hilbert space *.* There is a unique pair of mappings

*P* : *→ M* and *Q* : *→* such that *x* =*Px*+*Qx* for all

*x ∈.* Furthermore, *P* and *Q* have the following additional properties:

*(a) x ∈ M ⇒ Px* = *x and Qx* = 0*.*

*(b) x ∈ ⇒ Px* = 0 *and Qx* = *x.*

*(c) Px* is the closest vector in *M* to *x.*

*(d) Qx* is the closest vector in *to x.*

*(e)*  for all *x.*

*(f) P* and *Q* are linear maps.

 *MacCluer,* p. 12.

Proposition 1.23

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Theorem 1.24

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Orthogonality 1.21

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Proposition 1.22

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*Corollary 1.25:* If *M* is a closed, proper, subspace of *,* then there exists a nonzero vector *y* inwith *y ⊥ M.*

 *MacCluer,* p. 15.

*Linear Functional 1.26:* If *X* is a normed linear space over , a *linear functional* on *X* is a map Λ : *X →* satisfying Λ(α*x*+β*y*) =αΛ(*x*)+βΛ(*y*) for all vectors *x* and *y* in *X* and all scalars α and β .

 *MacCluer,* p. 15.

*Bounded Linear Functional 1.27:* A *bounded linear functional* on a normed linear space *X* is a linear

functional Λ : *X →* for which there exists a finite constant *C* satisfying ≤ *C* for all *x ∈ X*.

 *MacCluer,* p. 16.

*Proposition 1.28:* If *X* is a normed linear space, and

Λ : *X →* C is a linear functional,

then the following are equivalent:

(a)Λ is continuous.

(b)Λ is continuous at0*.*

(c)Λ is bounded.

 *MacCluer,* p. 16.

Bounded Linear Functional 1.27

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Proposition 1.28

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Corollary 1.25

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Linear Functional 1.26

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*Theorem 1.29:* Every bounded linear functionalΛ on a Hilbert spaceis given by inner product with a (unique) fixed vector in*:* Λ(*h*) = *h,.* Moreover, the norm of the linear functionalΛ is*.*

 *MacCluer,* p. 17.

*Lemma 1.30:* Let *P* : *→ M* be the orthogonal projection of a Hilbert space onto a closed subspace *M* of *.* We have = for allvectors *f* and *g* in *.*

 *MacCluer,* p. 18.

*Orthonormal Set 1.31:* An *orthonormal set* in a Hilbert space is a set with the properties:

(1) for every *e ∈* , *.* = 1, and

(2) for distinct vectors *e* and *f* in , = 0.

 *MacCluer,* p. 19.

*Orthonormal Basis 1.32:* An *orthonormal basis* for a Hilbert space is a maximal orthonormal set; that is, an orthonormal set that is not properly contained in any orthonormal set.

 *MacCluer,* p. 19.

Orthonormal Set 1.31

(in a Hilbert space)

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Orthonormal Basis 1.32

(in a Hilbert space)

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Theorem 1.29

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Lemma 1.30

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*Theorem 1.33:* If  is an orthonormal sequence in a Hilbert space , then the following conditions are equivalent:

(a) is an orthonormal basis.

(b) If *h ∈* and *h ⊥* for all *n,* then *h* = 0*.*

(c) For every *h ∈ , h* = ; equality here

 means the convergence in the norm ofof the

 partial sums to *h.*

(d)For every *h ∈,* there exist complex numbers so

 that *h* = *.*

(e) For every *h ∈ H ,*

(f) For all *h* and *g* in *H , .*

 *MacCluer,* p. 20.

Theorem 1.33

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