Normed Linear Space 1.1: Let X be a vector space over either the scalar field $\mathbb{R}$ of real numbers or the scalar field $\mathbb{C}$ of complex numbers. Suppose we have a function $\|\cdot\| : X \to [0,\infty)$ such that

(1) $\|x\| = 0$ if and only if $x = 0$, *(definiteness)*

(2) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, *(triangle inequality)* and

(3) $\|\alpha x\| = |\alpha| \, \|x\|$ for all scalars α and vectors x. *(homogeneity)*

MacCluer, p. 2.

Metric Space 1.9: A *metric space* is a set X with a function $d(\cdot, \cdot) : X \times X \to [0,\infty)$ satisfying, for $x, y,$ and z in X,

(1) $d(x,y) = 0$ if and only if $x = y$, (definiteness)

(2) $d(x,y) = d(y,x)$, and (symmetry)

(3) $d(x,y) + d(y, z) \geq d(x, z)$. (triangle inequality)

MacCluer, p. 5.

Cauchy Sequence 1.10: Let X be a metric space. A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if it has the following property: Given any $\varepsilon > 0$ there exists N such that if $n, m \geq N$, then $d(x_n, x_m) < \varepsilon$.

MacCluer, p. 5.

Complete Metric Space 1.11: A metric space is said to be *complete* if every Cauchy sequence in X converges in X.

MacCluer, p. 5.

Cauchy Sequence 1.10

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Normed Linear Space 1.1

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Complete Metric Space 1.11

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Metric Space 1.9

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Banach space 1.12: Let X be a normed linear space. If X is complete in the metric d defined from the norm by $d(x,y) = \|x - y\|$, we call X a *Banach space*.

MacCluer, p. 5.

Inner Product 1.13: Let X be a vector space over C. An *inner product* is a map $\langle\cdot,\cdot\rangle: X \times X \to \mathbb{C}$ satisfying, for x, y, and z in X and scalars $\alpha \in \mathbb{C}$,

(1) $\langle x, y\rangle = \langle\overline{y, x}\rangle$ for all x, y in X,
(hermitean, $\overline{x, y}$ denotes complex conjugation.)

(2) $\langle x, x\rangle \geq 0$, with $\langle x, x\rangle = 0$ (if and) only if $x = 0$,
(positive-definiteness)

(3) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, and

(4) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$. (3 & 4 together = sequilinearity)

MacCluer, p. 6.

Proposition 1.14: If $\langle\cdot,\cdot\rangle$ is an inner product on a vector space X, then for all x and y in X we have

$$|x, y|^2 \leq \langle x, x\rangle\langle y, y\rangle.$$

MacCluer, p. 7.

Proposition 1.15: If $\langle\cdot,\cdot\rangle$ is an inner product on a vector space X, then

$$\|x\| \equiv \langle x, x\rangle^{\frac{1}{2}}$$

MacCluer, p. 7.

Proposition 1.14

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Proposition 1.15

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Banach space 1.12

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Inner Product 1.13

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Hilbert space 1.16: A (complex) *Hilbert space* $\mathcal{H}$ is a vector space over $\mathbb{C}$ with an inner product such that $\mathcal{H}$ is complete in the metric

$$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{\frac{1}{2}} .$$

MacCluer, p. 8.

Proposition 1.18: If f is a analytic function in some closed disk $B(a,R)$, then

$$f(a) = \frac{1}{\pi R^2}\int_{B(a,R)} f\, dA.$$

MacCluer, p. 9.

Corollary 1.19: Fix $w \in \mathbb{D}$. For every $f \in L_a^2(D)$ we have

$$|f(w)| \leq \frac{1}{1-|w|}\|f\|_{L_a^2(D)} .$$

MacCluer, p. 9.

Theorem 1.20: The Bergman space $L_a^2(D)$ is a Hilbert space.

MacCluer, p. 10.

Corollary 1.19

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Hilbert space 1.16

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Theorem 1.20

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Proposition 1.18

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Orthogonality 1.21: Given vectors f, g in a Hilbert space $\mathcal{H}$, we say that f is *orthogonal* to g, written $f \perp g$, if $\langle f, g \rangle = 0$. For sets A and B in $\mathcal{H}$ we write $A \perp B$ if $\langle f, g \rangle = 0$ for all $f \in A$ and $g \in B$. Finally, $A^{\perp}$ is the set of all vectors $f \in \mathcal{H}$ such that $f \perp g$ for all g in A; for any set A this is always a subspace of $\mathcal{H}$, moreover since $A^{\perp} = \bigcap_{a \in A} \{a\}^{\perp}$, $A^{\perp}$ is a closed subspace by continuity of the inner product (see Exercise 1.8).

MacCluer, p. 11.

Proposition 1.22: If $f_1, f_2, \cdots, f_n$, are pairwise orthogonal vectors in a Hilbert
space, then
$\|f_1 + f_2 + \cdots + f_n\|^2 = \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_n\|^2$.

MacCluer, p. 11.

Proposition 1.23: Every nonempty, closed convex set K in a Hilbert space $\mathcal{H}$ contains a unique element of smallest norm. Moreover, given any $h \in \mathcal{H}$, there is a unique k_0 in K such that
$\|h - k_0\| = dist(h, K) \equiv inf\{\|h - k\| : k \in K\}$.

MacCluer, p. 12.

Theorem 1.24: Let M be a closed subspace of a Hilbert space $\mathcal{H}$. There is a unique pair of mappings $P : \mathcal{H} \to M$ and $Q : \mathcal{H} \to M^{\perp}$ such that $x = Px + Qx$ for all $x \in \mathcal{H}$. Furthermore, P and Q have the following additional properties:

(a) $x \in M \Rightarrow Px = x$ *and* $Qx = 0$.
(b) $x \in M^{\perp} \Rightarrow Px = 0$ *and* $Qx = x$.
(c) Px is the closest vector in M to x.
(d) Qx is the closest vector in $M^{\perp}$ *to* x.
(e) $\|Px\|^2 + \|Qx\|^2 = \|x\|^2$ for all x.
(f) P and Q are linear maps.

MacCluer, p. 12.

Proposition 1.23

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Theorem 1.24

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Orthogonality 1.21

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Proposition 1.22

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Corollary 1.25: If M is a closed, proper, subspace of $\mathcal{H}$, then there exists a nonzero vector y in $\mathcal{H}$ with $y \perp M$.

MacCluer, p. 15.

Linear Functional 1.26: If X is a normed linear space over $\mathbb{C}$, a *linear functional* on X is a map $\Lambda : X \to \mathbb{C}$ satisfying $\Lambda(\alpha x + \beta y) = \alpha\Lambda(x) + \beta\Lambda(y)$ for all vectors x and y in X and all scalars α and β.

MacCluer, p. 15.

Bounded Linear Functional 1.27: A *bounded linear functional* on a normed linear space X is a linear functional $\Lambda : X \to \mathbb{C}$ for which there exists a finite constant C satisfying $|\Lambda(x)| \leq C\|x\|$ for all $x \in X$.

MacCluer, p. 16.

Proposition 1.28: If X is a normed linear space, and $\Lambda : X \to \mathrm{C}$ is a linear functional,
then the following are equivalent:

(a) Λ is continuous.

(b) Λ is continuous at 0.

(c) Λ is bounded.

MacCluer, p. 16.

Bounded Linear Functional 1.27

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Proposition 1.28

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Corollary 1.25

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Linear Functional 1.26

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Theorem 1.29: Every bounded linear functional Λ on a Hilbert space $\mathcal{H}$ is given by inner product with a (unique) fixed vector h_0 in $\mathcal{H}: \Lambda(h) = h, h_0$. Moreover, the norm of the linear functional Λ is $\|h_0\|$.

MacCluer, p. 17.

Orthonormal Set 1.31: An *orthonormal set* in a Hilbert space $\mathcal{H}$ is a set $\mathcal{E}$ with the properties:

(1) for every $e \in \mathcal{E}$, $\|e\|. = 1$, and

(2) for distinct vectors e and f in $\mathcal{E}$, $\langle e, f \rangle = 0$.

MacCluer, p. 19.

Lemma 1.30: Let $P: \mathcal{H} \to M$ be the orthogonal projection of a Hilbert space $\mathcal{H}$ onto a closed subspace M of $\mathcal{H}$. We have $\langle f, Pg \rangle = \langle Pf, g \rangle$ for all vectors f and g in $\mathcal{H}$.

MacCluer, p. 18.

Orthonormal Basis 1.32: An *orthonormal basis* for a Hilbert space $\mathcal{H}$ is a maximal orthonormal set; that is, an orthonormal set that is not properly contained in any orthonormal set.

MacCluer, p. 19.

Orthonormal Set 1.31
(in a Hilbert space)

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Orthonormal Basis 1.32
(in a Hilbert space)

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Theorem 1.29

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Lemma 1.30

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Theorem 1.33: If $\{e_n\}_1^\infty$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, then the following conditions are equivalent:

(a) $\{e_n\}_1^\infty$ is an orthonormal basis.

(b) If $h \in \mathcal{H}$ and $h \perp e_n$ for all n, then $h = 0$.

(c) For every $h \in \mathcal{H}$, $h = \sum_1^\infty \langle h, e_n \rangle e_n$; equality here means the convergence in the norm of $\mathcal{H}$ of the partial sums to h.

(d) For every $h \in \mathcal{H}$, there exist complex numbers a_n so that $h = \sum_1^\infty a_n e_n$.

(e) For every $h \in H$, $\sum_1^\infty |\langle h, e_n \rangle|^2 = \|h\|^2$.

(f) For all h and g in H, $\sum_1^\infty \langle h, e_n \rangle \langle e_n, g \rangle = \langle h, g \rangle$.

MacCluer, p. 20.

Theorem 1.33

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