

*Field:* A ring  $R$  with identity  $1$ , where  $1 \neq 0$ , is called a division ring (or skew field) if every nonzero element  $a \in R$  has a multiplicative inverse, i.e., there exists  $b \in R$  such that  $ab = ba = 1$ . A commutative division ring is called a field. *D & F, p. 224.*

*field characteristic proposition 1\*:* The characteristic of a field  $F$ ,  $\text{ch}(F)$ , is either 0 or a prime  $p$ . If  $\text{ch}(F) = p$  then for any  $a \in F$ ,

$$p \cdot a = \underbrace{a + a + \cdots + a}_{p \text{ times}} = 0.$$

*D & F, p. 510.*

The *characteristic of a field  $F$* , denoted  $\text{ch}(F)$ , is defined to be the smallest positive integer  $p$  such that  $p \cdot 1_F = 0$  if such a  $p$  exists and is defined to be 0 otherwise. *D & F, p. 510.*

The *prime subfield* of a field  $F$  is the subfield of  $F$  generated by the multiplicative identity  $1_F$  of  $F$ . It is (isomorphic to) either  $\mathbb{Q}$  (if  $\text{ch}(F) = 0$ ) or  $F_p$  (if  $\text{ch}(F) = p$ ). *D & F, p. 511.*

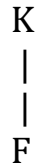
field

field characteristic proposition 1\*

field characteristic

prime subfield

If  $K$  is a field containing the sub field  $F$ , then  $K$  is said to be an *extension field* (or simply an *extension*) of  $F$ , denoted  $K / F$  or by the diagram



In particular, every field  $F$  is an extension of its prime sub field. The field  $F$  is sometimes called the *base field* of the extension. *D & F, p. 511.*

The *degree* (or *relative degree* or *index*) of a field extension  $K/F$ , denoted  $[K : F]$ , is the dimension of  $K$  as a vector space over  $F$  (i.e.,  $[K : F] = \dim_F K$ ). The extension is said to be *finite* if  $[K : F]$  is finite and is said to be *infinite* otherwise. *D & F, p. 512.*

*field isomorphism proposition 2\**: Let  $\varphi : F \rightarrow F'$  be a homomorphism of fields. Then  $\varphi$  is either identically 0 or is injective, so that the image of  $\varphi$  is either 0 or isomorphic to  $F$ . *D & F, p. 512.*

*isomorphic field theorem 3\**: Let  $F$  be a field and let  $p(x) \in F[x]$  be an irreducible polynomial. Then there exists a field  $K$  containing an isomorphic copy of  $F$  in which  $p(x)$  has a root. Identifying  $F$  with this isomorphic copy shows that there exists an extension of  $F$  in which  $p(x)$  has a root. *D & F, p. 512.*