Module: Let R be a ring (not necessarily commutative or with 1). A left R-module or a left module over R is a set M together with

1) a binary operation + on M under which M is an abelian group, and
2) an action of R on M (that is, a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies
   (a) $(r + s)m = rm + sm$, for all $r, s \in R$, $m \in M$,
   (b) $(rs)m = r(sm)$, for all $r, s \in R$, $m \in M$, and
   (c) $r(m + n) = rm + rn$, for all $r \in R$, $m, n \in M$.

If the ring R has a 1 we impose the additional axiom:
   (d) $1m = m$, for all $m \in M$. *D & F,* p. 337.

Modules over a field F and vector spaces over F are the same.

D & F, p. 337.

Submodule: Let R be a ring and let M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of ring elements, i.e., $rn \in N$, for all $r \in R$, $n \in N$.

D & F, p. 337.

Proposition 1: Let R be a ring and let M be an R-module. A subset N of M is a submodule of M if and only if
(1) $N \neq \emptyset$, and
(2) $x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.

D & F, p. 342.

module and vector space equivalence*

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module

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proposition 1

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submodule

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