*R-algebra:* Let R be a commutative ring with identity. An R-algebra is a ring A with identity together with a ring homomorphism f : R A mapping to such that the

subring f (R) of A is contained in the center of A.

*D & F,*  p. 342.

If A and B are two R-algebras, an *R-algebra homo- morphism (or isomorphism)* is a ring homomorphism (isomorphism, respectively) : A B mapping

to such that (r · a) = r · (a) for all r  R and a A.

*D & F,*  p. 343.

*Definitions.* Let R be a ring and let M and N be R-modules.

(1) A map (: MN is an R-module homomorphism if it

respects the R-module structures of M and N, i.e.,

(a) (x + y) = (x) + (y), for all x, y M and

(b) (rx) = r(x), for all r R, x M.

(2) An R-module homomorphism is an isomorphism (of R-modules) if it is both injective and surjective. The modules M and N are said to be isomorphic,

denoted MN, if there is some R-module isomorphism : MN.

*D & F,*  p. 345.

*R-module homomorphism and isomorphism definitions*

*continued*

(3) If : MN is an R -module homomorphism, let

ker  = {m M | (m) =0} (the kernel of ) and let (M) = {n N | n = (m) for some m M} (the

image of , as usual).

(4) Let M and N be R-modules and define HomR (M, N) to be the set of all R-module homomorphisms from M into N .

*D & F,*  p. 345.

R-module homomorphism (isomorphism)

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R-module homomorphism (isomorphism)

*continued*

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R-algebra

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R-algebra homomorphism (isomorphism)

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*Proposition 2:* Let M, N and L be R-modules.

(1) A map : M N is an R-module homomorphism if

and only if (rx + y) = r(x) + (y) for all x, y M

and all r R.

(2) Let ,  be elements of HomR (M, N). Define + by

( + )(m) = (m) + (m) for all m  M.

Then + HomR(M, N) and with this operation

HomR(M, N) is an abelian group. If R is a

commutative ring then for r R define r by

(r) (m) = r ((m)) for all m M.

*D & F,* p. 346.

*Proposition 2 cont.:*

(2) cont. Then r HomR(M, N) and with this action of

the commutative ring R the abelian group

HomR(M, N) is an R-module.

(3) If  HomR(L, M) and HomR(M, N) then

o  HomR(L, N).

(4) With addition as above and multiplication defined as

function composition, HomR(M, M) is a ring with 1 .

When R is commutative HomR(M, M) is an R-algebra.

*D & F,* p. 347.

*Endomorphism Ring:* The ring HomR(M, M) is called the *endomorphism ring* *of M* and will often be denoted by EndR(M) , or just End(M) when the ring R is clear from the context. Elements of End(M) are called *endomorphisms.*

*D & F,* p. 347.

*Proposition 3:* Let R be a ring, let M be an R-module and let N be a submodule of M . The (additive, abelian) quotient group M/N can be made into an R -module by definingan action of elements of R by

r(x + N) = (rx) + N, for all r R, x + N M/N.

The natural projection map : M M/N defined by

(x) = x + N is an R-module homomorphism with kernel N.

*D & F,* p. 348.

endomorphism ring

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proposition 3

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proposition 2

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proposition 2 cont.

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*Sum of 2 submodules:* Let A , B be submodules of the R-module M. The sum of A and B is the set

A + B = {a + b | a A , b B } . *D & F,* p. 349.

Module Isomorphism Theorems

(1) (*First Isomorphism Theorem for Modules*) Let M, N be R-modules and let : M N be an R-module homomorphism. Then ker is a submodule of M, and M/ker (M).

(2) (*Second Isomorphism Theorem*) Let A , B be submodules of the R-module M.

Then (A + B)/B A/(A B ) .

(3) (*Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with A B. Then (M/A)/(B/A) M/B. *D & F,* p. 349.

Module Isomorphism Theorems cont.

(4) (*Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by

A A/N, for all A N. This correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N). *D & F,* p. 349.

Definition. Let M be an R-module and let . . . . , be submodules of M. (*D & F,* p. 351)

(1) The sum of . . . . , is the set of all finite sums of elements from the sets : { | ; for all i}. Denote this sum by . . . . , .

(2) For any subset A of M let RA = { + | R, , m }

(where by convention RA = {O} if A = ). lf A is the finite set {} we shall write + for RA. Call RA the submodule of M generated by A.

module isomorphism theorems

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submodule definitions

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sum of 2 submodules

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module isomorphism theorems

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If N is a submodule of M (possibly N = M) and N = RA, for some subset A of M, we call A a set of generators or generating set for N, and we say N is generated by A.

(3) A submodule N of M (possibly N = M) is finitely generated if there is some finite subset A of M such that N = RA, that is, if N is generated by some finite subset.

(4) A submodule N of M (possibly N = M) is cyclic if there exists an element a M such that N = Ra, that is, if N is generated by one element: N = Ra = {ra | r R}.

*D & F,* p. 351.

*direct product of modules:* Let be a collection of R-modules. The collection of k-tuples () where with addition and action of R defined componentwise is called the direct product of , denoted .

*D & F,* p. 353.

*proposition 5:* Let be submodules of the R-module M. Then the following are equivalent:

(1) The map :  · · ·  · · · defined by

() =  · · · is an isomorphism (of

R-modules): · · · · · · .

(2) = 0 for all

j {1 , 2, . . . , k} .

(3) Every x · · · can be written uniquely in

the form · · · with .

*D & F,* p. 353.

Definition. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements of R and unique in A such that x = + , for some n . In this situation we say A is a basis or set of free generators for F. If R is a commutative ring the cardinality of A is called the rank of F (cf. Exercise 27).

*D & F,* p. 354.

proposition 5

R-module *free* on one of its subsets

submodule definitions cont.

direct product of modules

*Theorem 6:* For any set A there is a free R-module F(A) on the set A and F(A) satisfies the following universal property: if M is any R-module and : A  M is any map of sets, then there is a unique R-module homomorphism : F(A) M such that (a) = (a), for all a A , that is, the following diagram commutes.

*D & F,* p. 354.

*free R-module theorem 6\* cont.:*

A F(A)

M

When A is the finite set {}, F(A) = R  R

· · · R . (Compare: Section 6.3, free groups.)

*D & F,* p. 354.

*corollary 7:*

(1) If F1 and F2 are free modules on the same set A, there is a unique isomorphism between F1 and F2 which is the identity map on A.

(2) If F is any free R-module with basis A, then F F(A) . In particular, F enjoys the same universal property with respect to A as F (A) does in free R-module theorem.

*D & F,* p. 355.

*Theorem 8:* Let R be a subring of S, let N be a left R-module and let : N S N be the R-module homomorphism defined by (n) = 1n . Suppose that L is any left S-module (hence also an R-module) and that : N L is an R-module homomorphism from N to L. Then there is a unique S-module homomorphism

: S N L such that factors through , i.e.,

= o and the diagram

*D & F,* p. 362.

corollary 7

theorem 8

theorem 6

theorem 6 cont.

*unique module homomorphism theorem 8\* cont:*

N S N

L

commutes. Conversely, if : S N  L is an S-module homomorphism then  = o is an R-module homomorphism from N to L. *D & F,* p. 362.

Corollary 9. let : N S N be the R-module homomorphism defined the *unique module homomorphism theorem\**. Then N / ker is the unique largest quotient of N that can be embedded in any S-module. In particular, N can be embedded as an R -submodule of some left S -module if and only if is injective (in which case N is isomorphic to the R -submodule (N) of the S-module S N).

*D & F,* p. 362.

*R-balanced*: Let M be a right R -module, let N be a left R -module and let L be an abelian group (written additively). A map : M N L is called *R-balanced* or

*middle linear with respect to R* if

(m1 + m2, n) = (m1, n) + (m2, n)

(m, n1 + n2) = (m, n1) + (m, n2)

(m, rn) = (mr, n)

for all m, m1 , m2  M, n, n1, n2 N, and r R.

*D & F,* p. 365.

Theorem 10. Suppose R is a ring with 1 , M is a right R-module, and N is a left R-module. Let M N be the tensor product of M and N over R and let : M  N

M N be the R-balanced map defined above.

(1) If : M N  L is any group homomorphism from M N to an abelian group L then the composite map

= o  is an R-balanced map from M N to L.

(2) Conversely, suppose L is an abelian group and

: M N L is any R-balanced map. Then there is a unique group homomorphism : M N L

such that factors through , i.e., = o as in (1).

*D & F,* p. 365

R-balanced

theorem 10

theorem 8 cont.

corollary 9

Equivalently, the correspondence  in the commutative diagram

M x N M N

L

establishes a bijection

*D & F,* p. 365.

Corollary 11. Suppose D is an abelian group and

': M x N D is an R-balanced map such that

(i) the image of ' generates D as an abelian group, and

(ii) every R-balanced map defined on M N factors

through ' as in Theorem 10.

Then there is an isomorphism f:  D of abelian groups with ' = f o .

*D & F,* p. 366.

*(S, R)-bimodule:* Let R and S be any rings with 1. An abelian group M is called an (S, R)-bimodule if M is a left S-module, a right R-module, and s(mr) = (sm)r for all

s  S, r R and m M.

*D & F,* p. 366.

*standard R-module structure on M* : Suppose M is a left (or right) R-module over the commutative ring R.Then the (R, R)-bimodule structure on M defined by letting the left and right R-actionscoincide, i.e., mr = rm for all

m  M and r R, will be called the *standard R -module*

*structure on M*.

*D & F,* p. 367.

(S, R)-bimodule

standard R-module structure on M

theorem 10 cont.

corollary 11

*R-bilinear:*  Let R be a commutative ring with 1 and let M, N, and L be left R -modules.

The map : M N L is called *R-bilinear*  if it is

R -linear in each factor, i.e., if

( + , n) = (, n) + (, n), and

(m, + ) = (m, ) + (m, )

for all m, , M, n, ,  N and , R.

*D & F,* p. 368.

Corollary 12. Suppose R is a commutative ring. Let M and N be two left R-modules and let be the tensor product of M and N over R, where M is given the standard R-module structure. Then is a left R-module with

r(m n) = (rm)  n = (mr) n = m (rn),

*D & F,* p. 368.

and the map : M N with t(m, n) = m n is an R-bilinear map. If L is any left R-module then there is a bijection

where the correspondence between and is given by the commutative diagram

L *D & F,* p. 368.

*tensor product theorem 13:* Let M, M' be right

R-modules, let N, N' be left R-modules, and suppose

: M M' and : N N' are R-module homomorphisms.

(1) There is a unique group homomorphism, denoted

by  , mapping M N,into M' N' such that

( )(m n) = (m) (n) for all m M and

n N.

(2) If M, M' are also (S, R)-bimodules for some ring S

and is also an S-module homomorphism, then

is a homomorphism of left S-modules. In

particular, if R is commutative then f is

always an R -module homomorphism for the

standard R-module structures. *D & F,* p. 370.

corollary 12 cont.

tensor product theorem 13

R-bilinear

corollary 12

(3) If : M'  M" and : N' N" are R-module homomorphisms then

( ) o ( ) = ( o ) ( o ).

*associativity of the tensor product theorem 14\*:* Suppose M is a right R-module, N is an (R, T)-bimodule, and L is a left T-module. Then there is a unique isomorphism

(M ®R N) ®T L ;:;:: M ®R (N ®T L)

of abelian groups such that (m ® n) ® l 􀀭 m ® (n ® l). If M is an (S, R)-bimodule.

then this is an isomorphism of S-modules.