*R-algebra:* Let R be a commutative ring with identity. An R-algebra is a ring A with identity together with a ring homomorphism f : R $\rightarrow $ A mapping $1\_{R}$ to $1\_{A}$ such that the

subring f (R) of A is contained in the center of A.

 *D & F,*  p. 342.

If A and B are two R-algebras, an *R-algebra homo- morphism (or isomorphism)* is a ring homomorphism (isomorphism, respectively) $φ$: A $\rightarrow $ B mapping $1\_{A}$

to $1\_{B}$ such that $φ$(r · a) = r · $φ$(a) for all r $\in $ R and a $\in $ A.

 *D & F,*  p. 343.

*Definitions.* Let R be a ring and let M and N be R-modules.

(1) A map ($φ$: M$\rightarrow $N is an R-module homomorphism if it

respects the R-module structures of M and N, i.e.,

 (a) $φ$(x + y) = $φ$(x) + $φ$(y), for all x, y $\in $M and

 (b) $φ$(rx) = r$φ$(x), for all r $\in $ R, x $\in $ M.

(2) An R-module homomorphism is an isomorphism (of R-modules) if it is both injective and surjective. The modules M and N are said to be isomorphic,

denoted M$ ≅ $N, if there is some R-module isomorphism $φ$: M$\rightarrow $N.

 *D & F,*  p. 345.

*R-module homomorphism and isomorphism definitions*

*continued*

(3) If $φ$: M$\rightarrow $N is an R -module homomorphism, let

ker $φ$ = {m $\in $ M | $φ$(m) =0} (the kernel of $φ$) and let $φ$(M) = {n $\in $ N | n = $φ$(m) for some m $\in $ M} (the

image of $φ$, as usual).

(4) Let M and N be R-modules and define HomR (M, N) to be the set of all R-module homomorphisms from M into N .

 *D & F,*  p. 345.

R-module homomorphism (isomorphism)

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R-module homomorphism (isomorphism)

*continued*

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R-algebra

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R-algebra homomorphism (isomorphism)

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*Proposition 2:* Let M, N and L be R-modules.

(1) A map $φ$: M $\rightarrow $ N is an R-module homomorphism if

 and only if $φ$(rx + y) = r$φ$(x) + $φ$(y) for all x, y $\in $ M

 and all r $\in $ R.

(2) Let $φ$, $ψ$ be elements of HomR (M, N). Define $φ$ + $ψ$ by

 ($φ$ + $ψ$)(m) = $φ$(m) + $ψ$(m) for all m $\in $ M.

 Then $φ$ + $ψ$ $\in $ HomR(M, N) and with this operation

 HomR(M, N) is an abelian group. If R is a

 commutative ring then for r $\in $ R define r$φ$ by

 (r$φ$) (m) = r ($φ$(m)) for all m $\in $ M.

 *D & F,* p. 346.

*Proposition 2 cont.:*

(2) cont. Then r$φ$ $\in $ HomR(M, N) and with this action of

 the commutative ring R the abelian group

 HomR(M, N) is an R-module.

(3) If $φ$ $\in $ HomR(L, M) and $ψ\in $ HomR(M, N) then

 $ψ$ o $φ$ $\in $ HomR(L, N).

(4) With addition as above and multiplication defined as

 function composition, HomR(M, M) is a ring with 1 .

 When R is commutative HomR(M, M) is an R-algebra.

 *D & F,* p. 347.

*Endomorphism Ring:* The ring HomR(M, M) is called the *endomorphism ring* *of M* and will often be denoted by EndR(M) , or just End(M) when the ring R is clear from the context. Elements of End(M) are called *endomorphisms.*

 *D & F,* p. 347.

*Proposition 3:* Let R be a ring, let M be an R-module and let N be a submodule of M . The (additive, abelian) quotient group M/N can be made into an R -module by definingan action of elements of R by

r(x + N) = (rx) + N, for all r $\in $ R, x + N $\in $ M/N.

The natural projection map $π$: M $\rightarrow $ M/N defined by

 $π$(x) = x + N is an R-module homomorphism with kernel N.

 *D & F,* p. 348.

endomorphism ring

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proposition 3

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proposition 2

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proposition 2 cont.

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*Sum of 2 submodules:* Let A , B be submodules of the R-module M. The sum of A and B is the set

A + B = {a + b | a $\in $ A , b $\in $ B } . *D & F,* p. 349.

Module Isomorphism Theorems

(1) (*First Isomorphism Theorem for Modules*) Let M, N be R-modules and let $φ$: M $\rightarrow $ N be an R-module homomorphism. Then ker $φ$ is a submodule of M, and M/ker $φ$ $≅$ $φ$(M).

(2) (*Second Isomorphism Theorem*) Let A , B be submodules of the R-module M.

Then (A + B)/B $≅$ A/(A $⋂$ B ) .

(3) (*Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with A $⊆$ B. Then (M/A)/(B/A) $≅$ M/B. *D & F,* p. 349.

Module Isomorphism Theorems cont.

(4) (*Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by

A $\leftrightarrow $ A/N, for all A $⊇$ N. This correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N). *D & F,* p. 349.

Definition. Let M be an R-module and let $N\_{1}, N\_{2}, $. . . . , $N\_{n}$ be submodules of M. (*D & F,* p. 351)

(1) The sum of $N\_{1}, N\_{2}, $. . . . , $N\_{n}$ is the set of all finite sums of elements from the sets $N\_{n}$: {$a\_{1}+ a\_{2}+…+ a\_{n}$ | $a\_{i}\in N\_{i}$; for all i}. Denote this sum by $N\_{1}+ N\_{2}+ $. . . . , $N\_{n}$.

(2) For any subset A of M let RA = {$r\_{1}a\_{1}+$ $r\_{2}a\_{2}+…$+ $r\_{m}a\_{m}$ | $r\_{1},…, r\_{m}\in $R, $a\_{1},…, a\_{m}\in A$, m $\in $ $Z^{+}$}

(where by convention RA = {O} if A = $∅$). lf A is the finite set {$a\_{1},…, a\_{m}$} we shall write $r\_{1}a\_{1}+$ $r\_{2}a\_{2}+…$+ $r\_{n}a\_{n}$ for RA. Call RA the submodule of M generated by A.

module isomorphism theorems

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submodule definitions

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sum of 2 submodules

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module isomorphism theorems

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If N is a submodule of M (possibly N = M) and N = RA, for some subset A of M, we call A a set of generators or generating set for N, and we say N is generated by A.

(3) A submodule N of M (possibly N = M) is finitely generated if there is some finite subset A of M such that N = RA, that is, if N is generated by some finite subset.

(4) A submodule N of M (possibly N = M) is cyclic if there exists an element a $\in $ M such that N = Ra, that is, if N is generated by one element: N = Ra = {ra | r $\in $ R}.

 *D & F,* p. 351.

*direct product of modules:* Let $M\_{1}, …, M\_{k} $be a collection of R-modules. The collection of k-tuples ($m\_{1}, …, m\_{k}$) where$ m\_{i}\in M\_{i}$ with addition and action of R defined componentwise is called the direct product of $M\_{1}, …, M\_{k}$, denoted $M\_{1}× … × M\_{k}$.

 *D & F,* p. 353.

*proposition 5:* Let $N\_{1}, …, N\_{k}$ be submodules of the R-module M. Then the following are equivalent:

(1) The map $π$: $N\_{1}$ $×$ · · · $×$ $N\_{k}$ $\rightarrow $ $N\_{1}$ $+$ · · · $+$ $N\_{k}$ defined by

 $π$($a\_{1},…, a\_{k}$) = $a\_{1}$ $+$ · · · $+$ $a\_{k}$ is an isomorphism (of

 R-modules): $N\_{1}$ $+$ · · · $+$ $N\_{k}$ $≅$ $N\_{1}$ $×$ · · · $×$ $N\_{k}$.

(2) $N\_{j} ⋂ \left(N\_{1} + · · · + N\_{j-1} +N\_{j+1} +…+ N\_{k}\right) $= 0 for all

 j $\in $ {1 , 2, . . . , k} .

(3) Every x $\in $ $N\_{1}$ $+$ · · · $+$ $N\_{k}$ can be written uniquely in

 the form $a\_{1}$ $+$ · · · $+$ $a\_{k}$ with $a\_{i}\in N\_{i}$.

 *D & F,* p. 353.

Definition. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements $r\_{1},…, r\_{n}$ of R and unique $a\_{1},…, a\_{n}$ in A such that x = $r\_{1}a\_{1}+$ $r\_{2}a\_{2}+…$+ $r\_{n}a\_{n}$, for some n $\in $ $Z^{+}$. In this situation we say A is a basis or set of free generators for F. If R is a commutative ring the cardinality of A is called the rank of F (cf. Exercise 27).

 *D & F,* p. 354.

proposition 5

R-module *free* on one of its subsets

submodule definitions cont.

direct product of modules

*Theorem 6:* For any set A there is a free R-module F(A) on the set A and F(A) satisfies the following universal property: if M is any R-module and $φ$: A $\rightarrow $ M is any map of sets, then there is a unique R-module homomorphism $Φ$: F(A) $\rightarrow $ M such that $Φ$(a) = $φ$(a), for all a $\in $ A , that is, the following diagram commutes.

 *D & F,* p. 354.

*free R-module theorem 6\* cont.:*

A $\rightarrow inclusion \rightarrow $ F(A)

 $\searrow $ $\downright $

 $φ$ $Φ$

 $\searrow $ $\downright $

 $\searrow $ $\downright $

 M

When A is the finite set {$a\_{1},…, a\_{n}$}, F(A) = R$a\_{1}$ $⊕$ R$a\_{2}$ $⊕$

· · · $⊕$ R$a\_{n}$ $≅$ $R^{n}$ . (Compare: Section 6.3, free groups.)

 *D & F,* p. 354.

*corollary 7:*

(1) If F1 and F2 are free modules on the same set A, there is a unique isomorphism between F1 and F2 which is the identity map on A.

(2) If F is any free R-module with basis A, then F $≅$ F(A) . In particular, F enjoys the same universal property with respect to A as F (A) does in free R-module theorem.

 *D & F,* p. 355.

*Theorem 8:* Let R be a subring of S, let N be a left R-module and let $ι$: N $\rightarrow $ S $⨂\_{R}$N be the R-module homomorphism defined by $ι$(n) = 1$⨂$n . Suppose that L is any left S-module (hence also an R-module) and that $φ$: N $\rightarrow $ L is an R-module homomorphism from N to L. Then there is a unique S-module homomorphism

$Φ$: S $⨂\_{R}$N $\rightarrow $ L such that $φ$ factors through $Φ$, i.e.,

$φ$ = $Φ$ o $ι$ and the diagram

 *D & F,* p. 362.

corollary 7

theorem 8

theorem 6

theorem 6 cont.

*unique module homomorphism theorem 8\* cont:*

 N $\rightarrow $ $ι$ $\rightarrow $ S $⨂\_{R}$N

 $\searrow $ $\downright $

 $φ$ $Φ$

$ \searrow $ $\downright $

 L

commutes. Conversely, if $Φ$: S $⨂\_{R}$N $\rightarrow $ L is an S-module homomorphism then $φ$ = $Φ$ o $ι$ is an R-module homomorphism from N to L. *D & F,* p. 362.

Corollary 9. let $ι$: N $\rightarrow $ S $⨂\_{R}$N be the R-module homomorphism defined the *unique module homomorphism theorem\**. Then N / ker $ι$ is the unique largest quotient of N that can be embedded in any S-module. In particular, N can be embedded as an R -submodule of some left S -module if and only if $ι$ is injective (in which case N is isomorphic to the R -submodule $ι$(N) of the S-module S $⨂\_{R}$N).

 *D & F,* p. 362.

*R-balanced*: Let M be a right R -module, let N be a left R -module and let L be an abelian group (written additively). A map $φ$: M $×$ N $\rightarrow $ L is called *R-balanced* or

*middle linear with respect to R* if

$φ$(m1 + m2, n) = $φ$(m1, n) + $φ$(m2, n)

$φ$ (m, n1 + n2) = $φ$ (m, n1) + $φ$ (m, n2)

$φ$ (m, rn) = $φ$ (mr, n)

for all m, m1 , m2 $\in $ M, n, n1, n2 $\in $ N, and r $\in $ R.

 *D & F,* p. 365.

Theorem 10. Suppose R is a ring with 1 , M is a right R-module, and N is a left R-module. Let M $⨂\_{R}$N be the tensor product of M and N over R and let $ι$: M $×$ N $\rightarrow $

M $⨂\_{R}$N be the R-balanced map defined above.

(1) If $Φ$: M $⨂\_{R}$N $\rightarrow $ L is any group homomorphism from M $⨂\_{R}$N to an abelian group L then the composite map

$φ$ = $Φ$ o $ι$ is an R-balanced map from M $×$ N to L.

(2) Conversely, suppose L is an abelian group and

$φ$: M $×$ N $\rightarrow $ L is any R-balanced map. Then there is a unique group homomorphism $Φ$: M $⨂\_{R}$N $\rightarrow $ L

such that $φ$ factors through $ι$, i.e., $φ$ = $Φ$ o $ι$ as in (1).

 *D & F,* p. 365

R-balanced

theorem 10

theorem 8 cont.

corollary 9

Equivalently, the correspondence $φ\leftrightarrow $ $Φ$ in the commutative diagram

 M x N$\rightarrow ι\rightarrow $ M $⨂\_{R}$N

 $\searrow $ $\downright $

 $φ$ $Φ$

 $\searrow $ $\downright $

 $\searrow $ $\downright $

 L

establishes a bijection

 $\left\{\begin{matrix}R-balanced maps \\φ: M×N \rightarrow L \end{matrix}\right\}\leftrightarrow \left\{\begin{matrix}group homomorphisms\\Φ: M ⨂\_{R}N \rightarrow L\end{matrix}\right\}.$

 *D & F,* p. 365.

Corollary 11. Suppose D is an abelian group and

$ι$': M x N $\rightarrow $ D is an R-balanced map such that

 (i) the image of $ι$' generates D as an abelian group, and

 (ii) every R-balanced map defined on M $×$ N factors

 through $ι$' as in Theorem 10.

Then there is an isomorphism f: $M ⨂\_{R}N$ $≅$ D of abelian groups with $ι$' = f o $ι$.

 *D & F,* p. 366.

*(S, R)-bimodule:* Let R and S be any rings with 1. An abelian group M is called an (S, R)-bimodule if M is a left S-module, a right R-module, and s(mr) = (sm)r for all

s $\in $ S, r $\in $R and m $\in $M.

 *D & F,* p. 366.

*standard R-module structure on M* : Suppose M is a left (or right) R-module over the commutative ring R.Then the (R, R)-bimodule structure on M defined by letting the left and right R-actionscoincide, i.e., mr = rm for all

m $\in $ M and r $\in $ R, will be called the *standard R -module*

*structure on M*.

 *D & F,* p. 367.

(S, R)-bimodule

standard R-module structure on M

theorem 10 cont.

corollary 11

*R-bilinear:*  Let R be a commutative ring with 1 and let M, N, and L be left R -modules.

The map $φ$: M $×$ N $\rightarrow $ L is called *R-bilinear*  if it is

R -linear in each factor, i.e., if

 $φ$($r\_{1}m\_{1}$ + $r\_{2}m\_{2}$, n) = $r\_{1}φ$($m\_{1}$, n) + $r\_{2}φ$($m\_{2}$, n), and

 $φ$(m, $r\_{1}n\_{1}$ + $r\_{2}n\_{2}$) = $r\_{1}φ$(m, $n\_{1}$) + $r\_{2}φ$(m, $n\_{2}$)

for all m, $m\_{1}$, $m\_{2}$ $\in $ M, n, $n\_{1}$, $n\_{2}$ $\in $ N and $r\_{1}$, $r\_{2}$ $\in $ R.

 *D & F,* p. 368.

Corollary 12. Suppose R is a commutative ring. Let M and N be two left R-modules and let $M ⨂\_{R}N$ be the tensor product of M and N over R, where M is given the standard R-module structure. Then $M ⨂\_{R}N$ is a left R-module with

 r(m $⨂$ n) = (rm) $⨂$ n = (mr) $⨂$ n = m $⨂$ (rn),

 *D & F,* p. 368.

and the map $ι$: M $×$ N $\rightarrow $ $M ⨂\_{R}N$ with t(m, n) = m $⨂$ n is an R-bilinear map. If L is any left R-module then there is a bijection

 $\left\{\begin{matrix}R-bilinear maps \\φ: M × N \rightarrow L\end{matrix}\right\}\leftrightarrow \left\{\begin{matrix}R-module homomorphisms \\Φ: M ⨂\_{R}N \rightarrow L\end{matrix}\right\}$

where the correspondence between $φ$ and $Φ$ is given by the commutative diagram

 $M × N\rightarrow ι\rightarrow M ⨂\_{R}N $

 $\searrow $ $\downright $

 $φ$ $Φ$

 $\searrow $ $\downright $

 L *D & F,* p. 368.

*tensor product theorem 13:* Let M, M' be right

R-modules, let N, N' be left R-modules, and suppose

$φ$: M$ \rightarrow $ M' and $ψ$: N $\rightarrow $ N' are R-module homomorphisms.

 (1) There is a unique group homomorphism, denoted

 by $φ$ $⨂$ $ψ$, mapping M $⨂\_{R}$N,into M' $⨂\_{R} $N' such that

 ($φ$ $⨂$ $ψ$)(m $⨂$ n) = $φ$(m) $⨂$ $ψ$(n) for all m $\in $ M and

 n $\in $ N.

 (2) If M, M' are also (S, R)-bimodules for some ring S

 and $φ$ is also an S-module homomorphism, then

 $φ$ $⨂$ $ψ$ is a homomorphism of left S-modules. In

 particular, if R is commutative then $φ$ $⨂$ $ψ$ f is

 always an R -module homomorphism for the

 standard R-module structures. *D & F,* p. 370.

corollary 12 cont.

tensor product theorem 13

R-bilinear

corollary 12

(3) If $λ$: M' $\rightarrow $ M" and $μ$: N' $\rightarrow $ N" are R-module homomorphisms then

($λ$ $⨂$ $μ$) o ($φ$ $⨂$ $ψ$) = ($λ$ o $φ$) $⨂$ ($μ$ o $ψ$).

*associativity of the tensor product theorem 14\*:* Suppose M is a right R-module, N is an (R, T)-bimodule, and L is a left T-module. Then there is a unique isomorphism

(M ®R N) ®T L ;:;:: M ®R (N ®T L)

of abelian groups such that (m ® n) ® l 􀀭 m ® (n ® l). If M is an (S, R)-bimodule.

then this is an isomorphism of S-modules.