R-algebra: Let R be a commutative ring with identity. An R-algebra is a ring A with identity together with a ring homomorphism $f : R \to A$ mapping 1_R to 1_A such that the subring f (R) of A is contained in the center of A.

D & F, p. 342.

Definitions. Let R be a ring and let M and N be R-modules.

(1) A map (φ: M→N is an R-module homomorphism if it respects the R-module structures of M and N, i.e.,

(a) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all x, y ∈ M and

(b) $\varphi(rx) = r\varphi(x)$, for all r ∈ R, x ∈ M.

(2) An R-module homomorphism is an isomorphism (of R-modules) if it is both injective and surjective. The modules M and N are said to be isomorphic, denoted $M \cong N$, if there is some R-module isomorphism φ: M→N.

D & F, p. 345.

If A and B are two R-algebras, an *R-algebra homomorphism (or isomorphism)* is a ring homomorphism (isomorphism, respectively) $\varphi: A \to B$ mapping 1_A to 1_B such that $\varphi(r \cdot a) = r \cdot \varphi(a)$ for all r ∈ R and a ∈ A.

D & F, p. 343.

R-module homomorphism and isomorphism definitions continued

(3) If φ: M→N is an R -module homomorphism, let $\ker \varphi = \{m \in M \mid \varphi(m) = 0\}$ (the kernel of φ) and let $\varphi(M) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}$ (the image of φ, as usual).

(4) Let M and N be R-modules and define $\mathrm{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M into N .

D & F, p. 345.

R-module homomorphism (isomorphism)

R-algebra

R-module homomorphism (isomorphism)
continued

R-algebra homomorphism (isomorphism)

Proposition 2: Let M, N and L be R-modules.

(1) A map φ: M → N is an R-module homomorphism if and only if $\varphi(rx + y) = r\varphi(x) + \varphi(y)$ for all x, y ∈ M and all r ∈ R.

(2) Let φ, ψ be elements of Hom_R (M, N). Define $\varphi + \psi$ by $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$ for all m ∈ M.
Then $\varphi + \psi \in Hom_R(M, N)$ and with this operation $Hom_R(M, N)$ is an abelian group. If R is a commutative ring then for r ∈ R define $r\varphi$ by $(r\varphi)(m) = r(\varphi(m))$ for all m ∈ M.

D & F, p. 346.

Endomorphism Ring: The ring $Hom_R(M, M)$ is called the *endomorphism ring of M* and will often be denoted by EndR(M) , or just End(M) when the ring R is clear from the context. Elements of End(M) are called *endomorphisms.*

D & F, p. 347.

Proposition 2 cont.:

(2) cont. Then $r\varphi \in Hom_R(M, N)$ and with this action of the commutative ring R the abelian group $Hom_R(M, N)$ is an R-module.

(3) If $\varphi \in Hom_R(L, M)$ and $\psi \in Hom_R(M, N)$ then $\psi \circ \varphi \in Hom_R(L, N)$.

(4) With addition as above and multiplication defined as function composition, $Hom_R(M, M)$ is a ring with 1 . When R is commutative $Hom_R(M, M)$ is an R-algebra.

D & F, p. 347.

Proposition 3: Let R be a ring, let M be an R-module and let N be a submodule of M . The (additive, abelian) quotient group M/N can be made into an R -module by defining an action of elements of R by
$r(x + N) = (rx) + N$, for all r ∈ R, x + N ∈ M/N.
The natural projection map π: M → M/N defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.

D & F, p. 348.

endomorphism ring

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proposition 2

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proposition 3

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proposition 2 cont.

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Sum of 2 submodules: Let A , B be submodules of the R-module M. The sum of A and B is the set
$A + B = \{a + b \mid a \in A, b \in B\}$. *D & F,* p. 349.

Module Isomorphism Theorems

(1) (*First Isomorphism Theorem for Modules*) Let M, N be R-modules and let φ: M → N be an R-module homomorphism. Then ker φ is a submodule of M, and M/ker $\varphi \cong \varphi(M)$.

(2) (*Second Isomorphism Theorem*) Let A , B be submodules of the R-module M.
Then $(A + B)/B \cong A/(A \cap B)$.

(3) (*Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$.
Then $(M/A)/(B/A) \cong M/B$. *D & F,* p. 349.

Module Isomorphism Theorems cont.

(4) (*Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by
$A \leftrightarrow A/N$, for all $A \supseteq N$. This correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N). *D & F,* p. 349.

Definition. Let M be an R-module and let $N_1, N_2, \ldots, N_n$ be submodules of M. (*D & F,* p. 351)

(1) The sum of $N_1, N_2, \ldots, N_n$ is the set of all finite sums of elements from the sets N_n: $\{a_1 + a_2 + \cdots + a_n \mid a_i \in N_i$; for all i$\}$. Denote this sum by $N_1 + N_2 + \ldots, N_n$.

(2) For any subset A of M let $RA = \{r_1a_1 + r_2a_2 + \cdots + r_ma_m \mid r_1, \ldots, r_m \in R, a_1, \ldots, a_m \in A, m \in \mathbb{Z}^+\}$
(where by convention $RA = \{0\}$ if $A = \emptyset$). If A is the finite set $\{a_1, \ldots, a_m\}$ we shall write $r_1a_1 + r_2a_2 + \cdots + r_na_n$ for RA. Call RA the submodule of M generated by A.

module isomorphism theorems

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sum of 2 submodules

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submodule definitions

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module isomorphism theorems

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If N is a submodule of M (possibly N = M) and N = RA, for some subset A of M, we call A a set of generators or generating set for N, and we say N is generated by A.

(3) A submodule N of M (possibly N = M) is finitely generated if there is some finite subset A of M such that N = RA, that is, if N is generated by some finite subset.

(4) A submodule N of M (possibly N = M) is cyclic if there exists an element a ∈ M such that N = Ra, that is, if N is generated by one element: N = Ra = {ra | r ∈ R}.

D & F, p. 351.

proposition 5: Let $N_1, \dots, N_k$ be submodules of the R-module M. Then the following are equivalent:

(1) The map $\pi: N_1 \times \cdots \times N_k \to N_1 + \cdots + N_k$ defined by $\pi(a_1, \dots, a_k) = a_1 + \cdots + a_k$ is an isomorphism (of R-modules): $N_1 + \cdots + N_k \cong N_1 \times \cdots \times N_k$.

(2) $N_j \cap \left(N_1 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_k\right) = 0$ for all $j \in \{1, 2, \dots, k\}$.

(3) Every $x \in N_1 + \cdots + N_k$ can be written uniquely in the form $a_1 + \cdots + a_k$ with $a_i \in N_i$.

D & F, p. 353.

direct product of modules: Let $M_1, \dots, M_k$ be a collection of R-modules. The collection of k-tuples $(m_1, \dots, m_k)$ where $m_i \in M_i$ with addition and action of R defined componentwise is called the direct product of $M_1, \dots, M_k$, denoted $M_1 \times \dots \times M_k$.

D & F, p. 353.

Definition. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements $r_1, \dots, r_n$ of R and unique $a_1, \dots, a_n$ in A such that $x = r_1 a_1 + r_2 a_2 + \cdots + r_n a_n$, for some $n \in \mathbb{Z}^+$. In this situation we say A is a basis or set of free generators for F. If R is a commutative ring the cardinality of A is called the rank of F (cf. Exercise 27).

D & F, p. 354.

proposition 5

submodule definitions cont.

R-module *free* on one of its subsets

direct product of modules

Theorem 6: For any set A there is a free R-module F(A) on the set A and F(A) satisfies the following universal property: if M is any R-module and φ: A → M is any map of sets, then there is a unique R-module homomorphism Φ: F(A) → M such that $\Phi(a) = \varphi(a)$, for all $a \in A$, that is, the following diagram commutes.

D & F, p. 354.

free R-module theorem 6 cont.:*

A → inclusion → F(A)
↘ ↓
φ Φ
↘ ↓
↘ ↓
M

When A is the finite set $\{a_1, \dots, a_n\}$, F(A) = R$a_1 \oplus$ R$a_2 \oplus \cdots \oplus$ R$a_n \cong R^n$. (Compare: Section 6.3, free groups.)

D & F, p. 354.

corollary 7:

(1) If F_1 and F_2 are free modules on the same set A, there is a unique isomorphism between F_1 and F_2 which is the identity map on A.

(2) If F is any free R-module with basis A, then F ≅ F(A) . In particular, F enjoys the same universal property with respect to A as F (A) does in free R-module theorem.

D & F, p. 355.

Theorem 8: Let R be a subring of S, let N be a left R-module and let ι: N → S $\otimes_R$N be the R-module homomorphism defined by $\iota(n) = 1 \otimes n$. Suppose that L is any left S-module (hence also an R-module) and that φ: N → L is an R-module homomorphism from N to L. Then there is a unique S-module homomorphism Φ: S $\otimes_R$N → L such that φ factors through Φ, i.e., $\varphi = \Phi \circ \iota$ and the diagram

D & F, p. 362.

corollary 7

theorem 8

theorem 6

theorem 6 cont.

unique module homomorphism theorem 8 cont:*

$$\begin{array}{ccc} N \to \iota \to & & S \otimes_R N \\ \searrow & & \downarrow \\ \varphi & & \Phi \\ & \searrow & \downarrow \\ & & L \end{array}$$

commutes. Conversely, if Φ: $S \otimes_R N \to L$ is an S-module homomorphism then $\varphi = \Phi \circ \iota$ is an R-module homomorphism from N to L. *D & F,* p. 362.

Corollary 9. let ι: $N \to S \otimes_R N$ be the R-module homomorphism defined the *unique module homomorphism theorem**. Then N / ker ι is the unique largest quotient of N that can be embedded in any S-module. In particular, N can be embedded as an R-submodule of some left S-module if and only if ι is injective (in which case N is isomorphic to the R-submodule $\iota(N)$ of the S-module $S \otimes_R N$).

D & F, p. 362.

R-balanced: Let M be a right R-module, let N be a left R-module and let L be an abelian group (written additively). A map φ: $M \times N \to L$ is called *R-balanced* or *middle linear with respect to R* if

$\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$

$\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$

$\varphi(m, rn) = \varphi(mr, n)$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

D & F, p. 365.

Theorem 10. Suppose R is a ring with 1, M is a right R-module, and N is a left R-module. Let $M \otimes_R N$ be the tensor product of M and N over R and let ι: $M \times N \to M \otimes_R N$ be the R-balanced map defined above.

(1) If Φ: $M \otimes_R N \to L$ is any group homomorphism from $M \otimes_R N$ to an abelian group L then the composite map $\varphi = \Phi \circ \iota$ is an R-balanced map from $M \times N$ to L.

(2) Conversely, suppose L is an abelian group and φ: $M \times N \to L$ is any R-balanced map. Then there is a unique group homomorphism Φ: $M \otimes_R N \to L$ such that φ factors through ι, i.e., $\varphi = \Phi \circ \iota$ as in (1).

D & F, p. 365

R-balanced

theorem 8 cont.

theorem 10

corollary 9

Equivalently, the correspondence $\varphi \leftrightarrow \Phi$ in the commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\iota} & M \otimes_R N \\ & \searrow^{\varphi} & \downarrow \Phi \\ & & L \end{array}$$

establishes a bijection

$$\left\{ \begin{array}{c} R-\text{balanced maps} \\ \varphi: M \times N \rightarrow L \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{group homomorphisms} \\ \Phi: M \otimes_R N \rightarrow L \end{array} \right\}.$$

D & F, p. 365.

Corollary 11. Suppose D is an abelian group and ι': M x N → D is an R-balanced map such that

(i) the image of ι' generates D as an abelian group, and

(ii) every R-balanced map defined on $M \times N$ factors through ι' as in Theorem 10.

Then there is an isomorphism f: $M \otimes_R N \cong D$ of abelian groups with $\iota' = f \circ \iota$.

D & F, p. 366.

(S, R)-bimodule: Let R and S be any rings with 1. An abelian group M is called an (S, R)-bimodule if M is a left S-module, a right R-module, and s(mr) = (sm)r for all $s \in S$, $r \in R$ and $m \in M$.

D & F, p. 366.

standard R-module structure on M: Suppose M is a left (or right) R-module over the commutative ring R. Then the (R, R)-bimodule structure on M defined by letting the left and right R-actions coincide, i.e., mr = rm for all $m \in M$ and $r \in R$, will be called the *standard R -module structure on M*.

D & F, p. 367.

(S, R)-bimodule

theorem 10 cont.

standard R-module structure on M

corollary 11

R-bilinear: Let R be a commutative ring with 1 and let M, N, and L be left R -modules.
The map φ: M × N → L is called *R-bilinear* if it is R -linear in each factor, i.e., if

$\varphi(r_1 m_1 + r_2 m_2, \mathrm{n}) = r_1\varphi(m_1, \mathrm{n}) + r_2\varphi(m_2, \mathrm{n})$, and

$\varphi(\mathrm{m}, r_1 n_1 + r_2 n_2) = r_1\varphi(\mathrm{m}, n_1) + r_2\varphi(\mathrm{m}, n_2)$

for all m, m_1, m_2 ∈ M, n, n_1, n_2 ∈ N and r_1, r_2 ∈ R.

D & F, p. 368.

Corollary 12. Suppose R is a commutative ring. Let M and N be two left R-modules and let $\mathrm{M} \otimes_R \mathrm{N}$ be the tensor product of M and N over R, where M is given the standard R-module structure. Then $\mathrm{M} \otimes_R \mathrm{N}$ is a left R-module with

r(m ⊗ n) = (rm) ⊗ n = (mr) ⊗ n = m ⊗ (rn),

D & F, p. 368.

and the map ι: M × N → $\mathrm{M} \otimes_R \mathrm{N}$ with t(m, n) = m ⊗ n is an R-bilinear map. If L is any left R-module then there is a bijection

$$\left\{\begin{matrix}\text{R} - \text{bilinear maps} \\ \varphi: \text{M} \times \text{N} \to \text{L}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{R} - \text{module homomorphisms} \\ \Phi: \text{M} \otimes_R \text{N} \to \text{L}\end{matrix}\right\}$$

where the correspondence between φ and Φ is given by the commutative diagram

$$\begin{array}{ccc} \text{M} \times \text{N} & \xrightarrow{\ \iota\ } & \text{M} \otimes_R \text{N} \\ & \searrow & \downarrow \\ & \varphi & \Phi \\ & \searrow & \downarrow \\ & & \text{L} \end{array}$$

D & F, p. 368.

tensor product theorem 13: Let M, M' be right R-modules, let N, N' be left R-modules, and suppose φ: M → M' and ψ: N → N' are R-module homomorphisms.

(1) There is a unique group homomorphism, denoted by $\varphi \otimes \psi$, mapping $\mathrm{M} \otimes_R \mathrm{N}$,into $\mathrm{M'} \otimes_R \mathrm{N'}$ such that $(\varphi \otimes \psi)(\mathrm{m} \otimes \mathrm{n}) = \varphi(\mathrm{m}) \otimes \psi(\mathrm{n})$ for all m ∈ M and n ∈ N.

(2) If M, M' are also (S, R)-bimodules for some ring S and φ is also an S-module homomorphism, then $\varphi \otimes \psi$ is a homomorphism of left S-modules. In particular, if R is commutative then $\varphi \otimes \psi$ f is always an R -module homomorphism for the standard R-module structures.

D & F, p. 370.

corollary 12 cont.

R-bilinear

tensor product theorem 13

corollary 12

(3) If λ: M' → M" and μ: N' → N" are R-module homomorphisms then
$(\lambda \otimes \mu)$ o $(\varphi \otimes \psi) = (\lambda$ o $\varphi) \otimes (\mu$ o $\psi)$.

associativity of the tensor product theorem 14:*
Suppose M is a right R-module, N is an (R, T)-bimodule, and L is a left T-module. Then there is a unique isomorphism
(M ®R N) ®T L ;:;:: M ®R (N ®T L)
of abelian groups such that (m ® n) ® l □ m ® (n ® l). If M is an (S, R)-bimodule.
then this is an isomorphism of S-modules.